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LETTER TO THE EDITOR

A note on dynamics of fluctuations near the Bénard instability

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Abstract. Effects of nonlinear terms in hydrodynamic equations on dynamical properties of fluctuations near the Bénard instability are discussed. When the boundary condition is symmetric with respect to the horizontal plane bisecting the liquid layer, nonlinearities are shown to have very little effect upon the dynamics of fluctuations.

Recently, the behaviour of hydrodynamic fluctuations near the Bénard instability was studied theoretically on the basis of linearized hydrodynamic equations with random forces for fluctuations (Zaitsev and Shliomis 1971, to be referred to as ZS). It was found that fluctuations are characterized by the correlation range r_c and the correlation time τ_c which increase indefinitely near the threshold as $|R-R_0|^{-1/2}$ and $|R-R_0|^{-1}$, respectively, where R is the Rayleigh number and R_0 is its threshold value. The authors note that very close to the threshold, nonlinear terms may become important and even the underlying hydrodynamic equations may cease to be valid. This is quite analogous to the behaviour of critical fluctuations near second order phase transitions (eg Kawasaki 1972, and the references quoted therein), where, indeed, nonlinear coupling among hydrodynamic fluctuations arising from drift terms often has decisive effects upon the dynamics of fluctuations which invalidate the usual hydrodynamic equations. Hence it is natural to inquire about possible effects of nonlinear terms in the Bénard instability as well. It is the purpose of this note to point out the importance of the symmetry of the boundary conditions on the nonlinear mode coupling.

We follow the notation of ZS and start from the following nonlinear Langevin-type equation in the dimensionless form:

$$\frac{\partial v_\alpha}{\partial t} = -\frac{\partial p}{\partial x_\alpha} + \Delta v_\alpha + \mathrm{D}\gamma_\alpha T - v_\beta \frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial s_{\alpha\beta}}{\partial x_\beta} \tag{1}$$

$$P\frac{\partial T}{\partial t} = \Delta T + \mathrm{D}\boldsymbol{\gamma} \,.\, \boldsymbol{v} - v_\alpha \frac{\partial T}{\partial x_\alpha} - \mathrm{div}\, \boldsymbol{q} \tag{2}$$

$$\mathrm{div}\, \boldsymbol{v} = 0, \tag{3}$$

where $\boldsymbol{v}$ and T are subject to the rigid or free boundary conditions (eg Chandrasekhar 1961). The linearized hydrodynamic equations without random forces corresponding to (1), (2) and (3) possess the complete system of eigenfunctions $\boldsymbol{v}_n(\boldsymbol{r})$, $T_n(\boldsymbol{r})$, $P_n(\boldsymbol{r})$

apart from a time factor $\exp(-t\lambda_n)$, which is normalized as

$$\int (\boldsymbol{v}_m \cdot \boldsymbol{v}_n + PT_mT_n)\,\mathrm{d}V = \delta_{mn}, \tag{4}$$

and in terms of which we expand the solutions of (1), (2) and (3) as

$$\boldsymbol{v}(t) = \sum_n c_n(t)\boldsymbol{v}_n, \tag{5}$$

etc. We then find that $c_n(t)$ satisfies the following equation:

$$\dot{c}_n = -\lambda_n c_n - \tfrac{1}{2}\sum_{lm} \mathscr{V}_{n,lm}c_lc_m + f_n \tag{6}$$

where f_n is the random force given by the right hand side of equation (14) of zs, and the mode coupling coefficient $\mathscr{V}_{n,lm}$ is given by

$$\mathscr{V}_{n,lm} \equiv \int \mathrm{d}V\left\{v_{n\alpha}^*\left(v_{l\beta}\frac{\partial v_{m\alpha}}{\partial x_\beta} + v_{m\beta}\frac{\partial v_{l\alpha}}{\partial x_\beta}\right) + PT_n^*\left(v_{l\alpha}\frac{\partial T_m}{\partial x_\alpha} + v_{m\alpha}\frac{\partial T_l}{\partial x_\alpha}\right)\right\}. \tag{7}$$

Regarding (6) as a kinetic equation for fluctuations $c_n(t)$, time correlation functions of hydrodynamic fluctuations can be readily found by the standard method (Kawasaki 1972).

Let us now investigate the properties of $\mathscr{V}_{n,lm}$ for the case of a flat horizontal layer of liquid with the same boundary conditions at the top and the bottom of the layer. The relevant eigenfunctions are those characterized by a quasi-continuous wavevector $\boldsymbol{k}$ in the plane of the layer and a discrete 'quantum number' ν given by

$$v_{n\alpha} = L^{-1}u_\nu^\alpha(z, \boldsymbol{k})\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}), \qquad T_n = L^{-1}T_\nu(z, \boldsymbol{k})\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}) \tag{8}$$

(7) then becomes

$$\begin{aligned}\mathscr{V}_{\nu\boldsymbol{q},\nu'\boldsymbol{k},\nu''\boldsymbol{q}-\boldsymbol{k}} = \frac{1}{L}\int_{-1/2}^{1/2} \mathrm{d}z\Big\{&\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{u}_{\nu''}(\boldsymbol{q}-\boldsymbol{k})(\boldsymbol{u}_\nu^*(\boldsymbol{q})\cdot\boldsymbol{u}_{\nu'}(\boldsymbol{k}) + u_\nu^{z*}(\boldsymbol{q})u_{\nu'}^z(\boldsymbol{k})) + u_{\nu''}^z(\boldsymbol{q}-\boldsymbol{k})\\ &\times\left(\boldsymbol{u}_\nu^*(\boldsymbol{q})\cdot\frac{\partial \boldsymbol{u}_{\nu'}(\boldsymbol{k})}{\partial z} + u_\nu^{z*}(\boldsymbol{q})\frac{\partial u_{\nu'}^z(\boldsymbol{k})}{\partial z}\right)\\ &+PT_\nu^*(\boldsymbol{q})\left(\mathrm{i}\boldsymbol{u}_{\nu''}(\boldsymbol{q}-\boldsymbol{k})\cdot\boldsymbol{k}T_{\nu'}(\boldsymbol{k}) + u_{\nu''}^z(\boldsymbol{q}-\boldsymbol{k})\frac{\partial T_{\nu'}(\boldsymbol{k})}{\partial z}\right)\Big\}\\ &+\text{ terms obtained by interchanging } \boldsymbol{k}, \nu' \text{ and } \boldsymbol{q}-\boldsymbol{k}, \nu'',\end{aligned} \tag{9}$$

where the argument z was omitted in u and T, and all the vectors in (9) are the two-dimensional ones lying in the plane of the layer whose boundaries are at $z = \pm\frac{1}{2}$.

By the nature of the problem the eigenfunctions u_ν^α and T_ν can be chosen to be either symmetric or antisymmetric with respect to the surface $z = 0$. Furthermore, inspection of the linearized hydrodynamic equations together with div $\boldsymbol{v} = 0$ shows that T_ν and u_ν^z have the same parity $\epsilon_\nu = \pm 1$ (ie symmetry of eigenfunctions with respect to the change of sign of z) which is opposite to that of u_ν^x and u_ν^y. Then, one can readily verify that each term in the integrand of (9) has the same parity $-\epsilon_\nu\epsilon_{\nu'}\epsilon_{\nu''}$. Thus $\mathscr{V}$ vanishes unless $\epsilon_\nu\epsilon_{\nu'}\epsilon_{\nu''} = -1$. Near the instability threshold only a particular branch of the modes $\nu = 0$ develops anomalous fluctuations (Zaitsev and Shliomis 1972), and in the particular boundary conditions under consideration it turns out that

$\epsilon_0 = 1$. All other modes need not be considered. The only relevant mode coupling coefficients (9) with $\nu = \nu' = \nu'' = 0$ then identically vanish. On the other hand, the mode coupling arising from the term

$$\tfrac{1}{2}\Phi^{1/2}\left(\frac{\rho\nu^5}{k_B T^2 \chi c_p l}\right)^{1/2}\left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha}\right)^2 \tag{10}$$

which we have ignored on the right hand side of (2) is of the dissipative nature, and hence gives only a negative contribution to the relaxation rate of anomalous fluctuations (Kawasaki 1973).

In this note we have shown that for the boundary conditions symmetric with respect to the horizontal plane bisecting the layer the mode coupling coefficients $\mathscr{V}$ among the relevant modes near the threshold vanish identically, and the zs results of linear theory are expected to hold up to the threshold except for possible elongation of characteristic times due to (10). On the other hand, in the absence of the symmetry, for instance, in the case with one rigid and one free boundary, this kind of cancellation of $\mathscr{V}$'s does not occur. We hope to report on this case on some other occasion.

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